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REPORT NO. 1920

CHARACTERIZATION OF SOLUTIONS OF THE EQUATION $e^{Ax} = x$.

P. R. Schlegel

August 1976

USA BALLISTIC RESEARCH LABORATORIES
ABERDEEN PROVING GROUND, MARYLAND

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER REPORT NO. 1920	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Characterization of Solutions of the Equation $e^{Ax} = x$.		5. TYPE OF REPORT & PERIOD COVERED Final
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) P. R. Schlegel		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS U.S. Army Ballistic Research Laboratories Aberdeen Proving Ground, MD 21005		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS RDTE Proj. No. 1W161102AH43
11. CONTROLLING OFFICE NAME AND ADDRESS USArmy Materiel Development & Readiness Command 5001 Eisenhower Avenue Alexandia, VA 22333		12. REPORT DATE AUGUST 1976
		13. NUMBER OF PAGES 21
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Characterize, Solution, Rolle's Theorem

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper characterizes the solutions of the equation $e^{Ax} = x$, where A is any real number.

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I. INTRODUCTION

This paper characterizes the solutions of the equation

$$x = e^{Ax}, \tag{1}$$

which arises in several areas of investigation, e. g., determining the maximum range of an artillary trajectory for fixed time under a given aerodynamic drag [1]; determining the residual velocity of a bullet traveling through a liquid as a function of fuze delay time and striking velocity [2]; calculation of the burning rate for constant frequency in a T-Burner experiment [3]; and in the problem of intense surface heating of a slab [4].

II. CHARACTERIZATION OF SOLUTIONS

We will now state and prove a series of lemmas, which will characterize the solutions of $e^{Ax} = x$ as a function of A. The first three lemmas will give the algorithms for generating the solutions. In the sequel a value of x which satisfies $e^{Ax} = x$ will be referred to as a solution. Also, log(x) will denote the natural logarithm of x.

<u>Lemma 1</u>. For $0 < A < e^{-1}$, the sequence $y_{i+1} = e^{Ay_i}$, $y_0 = e$, is a bounded monotone decreasing sequence and its limit is a solution.

Proof: $y_0 = e > e^{Ay_0} = y_1$. Assume $y_0 > y_1 > \ldots > y_i$. Then

---

[1] McCoy, H., Private consultation.

[2] Walther, R., et al, Forthcoming report concerning vulnerability analysis of MiG-21 aircraft to 20 mm and improved 20 mm projectiles with various types fuzing.

[3] Ibiricu, M. M., Forthcoming report on analysis of T-Burner results based on A-13 propellant data.

[4] Masters, J. I., "Problems of Intense Surface Heating of a Slab Accompanied by Change Phase," Journal of Applied Physics, Vol. 21, 1956, pp. 477-484.

$y_{i+1} = e^{Ay_i} < e^{Ay_{i-1}} = y_i$. Hence, by induction the sequence is monotone decreasing and bounded by zero; in fact, it is bounded below by e^A. Therefore, $y_i \to y$ and y is a solution.

<u>Lemma 2</u>. For $0 < A < e^{-1}$, the sequence $y_{i+1} = A^{-1}\log(y_i)$, $y_0 = A^{-1}$, is a bounded monotone increasing sequence and its limit is a solution.

Proof: For $0 < A < e^{-1}$, $y_1 = A^{-1}\log(A^{-1}) > A^{-1}\log(e) = A^{-1} = y_0$. Assume $y_0 < y_1 < \ldots < y_i$, then $y_{i+1} = A^{-1}\log(y_i) > A^{-1}\log(y_{i-1}) = y_i$. Thus, by induction the sequence is monotone increasing.

To show the sequence is bounded, we will generate a bound for the first three terms, assume a form of the bound on y_k, $k = 1, \ldots, i$, then use induction. To simplify notation, let $U = \log(A^{-1})$. Then

$$y_1 = A^{-1}U,$$

$$y_2 = A^{-1}\log(y_1) = A^{-1}[U + \log(U)],$$

$$y_3 = A^{-1}\log(y_2) = A^{-1}[U + \log(U + \log(U))]$$

$$= A^{-1}[U + \log(U) + \log(1 + U^{-1}\log(U))].$$

Since $\log(1 + x) < x$ for $x > 0$, then

$$y_3 < A^{-1}[U + \log(U)[1 + U^{-1}]].$$

Note that $U > 1$, this implies $\log(U) > 0$.

Assume the form of the bound holds for y_i, that is,

$$y_i < A^{-1}[U + \log(U)[1 + U^{-1} + \ldots + U^{2-i}]].$$

Then

$$y_{i+1} = A^{-1}\log(y_i)$$

$$< A^{-1}[\log(A^{-1}[U + \log(U)[1 + U^{-1} + \ldots + U^{2-i}]])]$$

$$< A^{-1}[U + \log(U + \log(U)[1 + U^{-1} + \ldots + U^{2-i}])]$$

$$< A^{-1}[U + \log(U) + \log(1 + \log(U)[U^{-1} + \ldots + U^{1-i}])]$$

$$< A^{-1}[U + \log(U)[1 + U^{-1} + \ldots + U^{1-i}]].$$

Thus, the same form of the bound holds for y_{i+1}. Now

$$\sum_{k=0}^{i} U^{-k} < \sum_{k=0}^{\infty} U^{-k} = \frac{U}{U - 1},$$

then for all i

$$y_i < A^{-1}[U + U(U - 1)^{-1}\log(U)].$$

Therefore, $y_i \to y$ which satisfies $A^{-1}\log(y)$ or $y = e^{Ay}$, that is, y is a solution.

Lemma 3. For $A < 0$, the sequence $y_{i+1} = e^{Ay_i}$, $y_0 = e$, has a unique limit and its limit is a solution.

Proof: By induction we will show the terms of even subscripts form a bounded monotone decreasing subsequence and the terms of odd subscripts form a bounded monotone increasing subsequence. Now

$$y_1 = e^{Ae} < e = y_0$$

$$y_2 = e^{Ay_1} > e^{Ay_0} = y_1,$$

that is,

$$y_1 < y_2 < e = y_0,$$

and

$$y_1 = e^{Ay_0} < e^{Ay_2} = y_3 = e^{Ay_2} < e^{Ay_1} = y_2.$$

Assume for i even

$$y_{i-1} < y_i < y_{i-2}.$$

Then

$$y_{i-1} = \exp(Ay_{i-2}) < \exp(Ay_i) = y_{i+1} = \exp(Ay_i) < \exp(Ay_{i-1}) = y_i.$$

For i odd assume

$$y_{i-2} < y_i < y_{i-1}.$$

Then

$$y_i = \exp(Ay_{i-1}) < \exp(Ay_i) = y_{i+1} = \exp(Ay_i) < \exp(Ay_{i-2}) = y_{i-1}.$$

Hence, by induction the sequence has the following ordering:

$$y_1 < y_3 < y_5 < \ldots < y_{2i+1} < \ldots < y_{2i} < \ldots < y_4 < y_2 < y_0.$$

Suppose $y_{2i+1} \to y_0$ and $y_{2i} \to y_E$, where y_0 and y_E are distinct. Then by Rolle's theorem, there exists $x_1 \in (y_0, y_E)$ such that $\frac{d}{dx}(e^{Ax} - x)|_{x=x_1} = 0$. This implies $\exp(Ax_1) = A^{-1} < 0$. This is a contradiction.

Lemma 4. For $A < 0$ there exists a unique solution.

Proof: This follows immediately from lemma 3 and Rolle's theorem.

Lemma 5. For $0 < A < e^{-1}$ there exist exactly two solutions.

Proof: From lemma 1 and lemma 2 there exist two distinct solutions, call them x_1 and x_2. Let $h(x) = e^{Ax} - x$. Suppose there exists a solution x_3 distinct from x_1 and x_2. Now $\frac{dh}{dx} = Ae^{Ax} - 1$, that is, $\frac{dh}{dx}$ vanishes only at $x = A^{-1}\log(A^{-1})$. This is a contradiction, since Rolle's theorem says $\frac{dh}{dx}$ must vanish for at least two distinct points.

Lemma 6. For $A > e^{-1}$ there exist no solutions.

Proof: Let $g(x) = e^{Ax}$ and $f(x) = x$. Now $g(x) > 1 + g'(0)x$ and $g'(0) > e^{-1}$. Then for a point, x_0, of intersection (a solution) of $f(x)$ and $g(x)$, $x_0 > 1.58$. For $x > 1.58$, $g(x) > g(1.58) + g'(1.58)(x - 1.58)$. This implies $x_0 > 4.86 > e$. Since $g(e) > e$ and since, for $x > e$,

$g'(x) > g'(e) > 1$, there exist no solutions.

<u>Lemma 7</u>. For $A = 0$ or $A = e^{-1}$, $e^{Ax} = x$ has a unique solution.

Proof: Trivially, $A = 0$ has the unique solution $x = 1$. For $A = e^{-1}$, $x = e$ is a solution. By assuming another solution $x_1 \neq e$, this results in an immediate contradiction to Rolle's theorem.

III. CONCLUSION

The previous seven lemmas have characterized the solutions of $e^{Ax} = x$. Figure 1 gives a graph of the solutions, and a tabular form of the solutions is given in Table 1.

There are more general forms of equations for which solutions are desired. By an appropriate change of variable, these can be readily reduced to the form $e^{Ax} = x$. For example, consider the solutions of

$$x + a = be^{c(x+d)}. \tag{2}$$

Let $z = (\frac{x + a}{b})e^{c(a-d)}$ or $x = e^{c(d-a)}bz - a$, then (2) reduces to $z = e^{Az}$, where $A = bce^{c(d-a)}$. Another form is

$$ax = \exp(bx^c), \tag{3}$$

then $x = a^{-1}z^{(c^{-1})}$, where z is a solution of $e^{Az} = z$ for $A = bca^{-c}$.

For $A < 0$, lemma 4 shows the existence of a unique solution. Lemma 3 gives a simple algorithm for approximating the solution, but for $A < -1$, this algorithm converges slowly. Fritsch, Shafer and Crowley [5] considered the case for $A < 0$ and gave an efficient algorithm for approximating the solution.

[5] Fritsch, F. N., Shafer, R. E. and Crowly, W. P., "Solution of Transcendental Equation $we^w = x$", Communication of the ACM, Vol. 16, 1973, pp. 123-124.

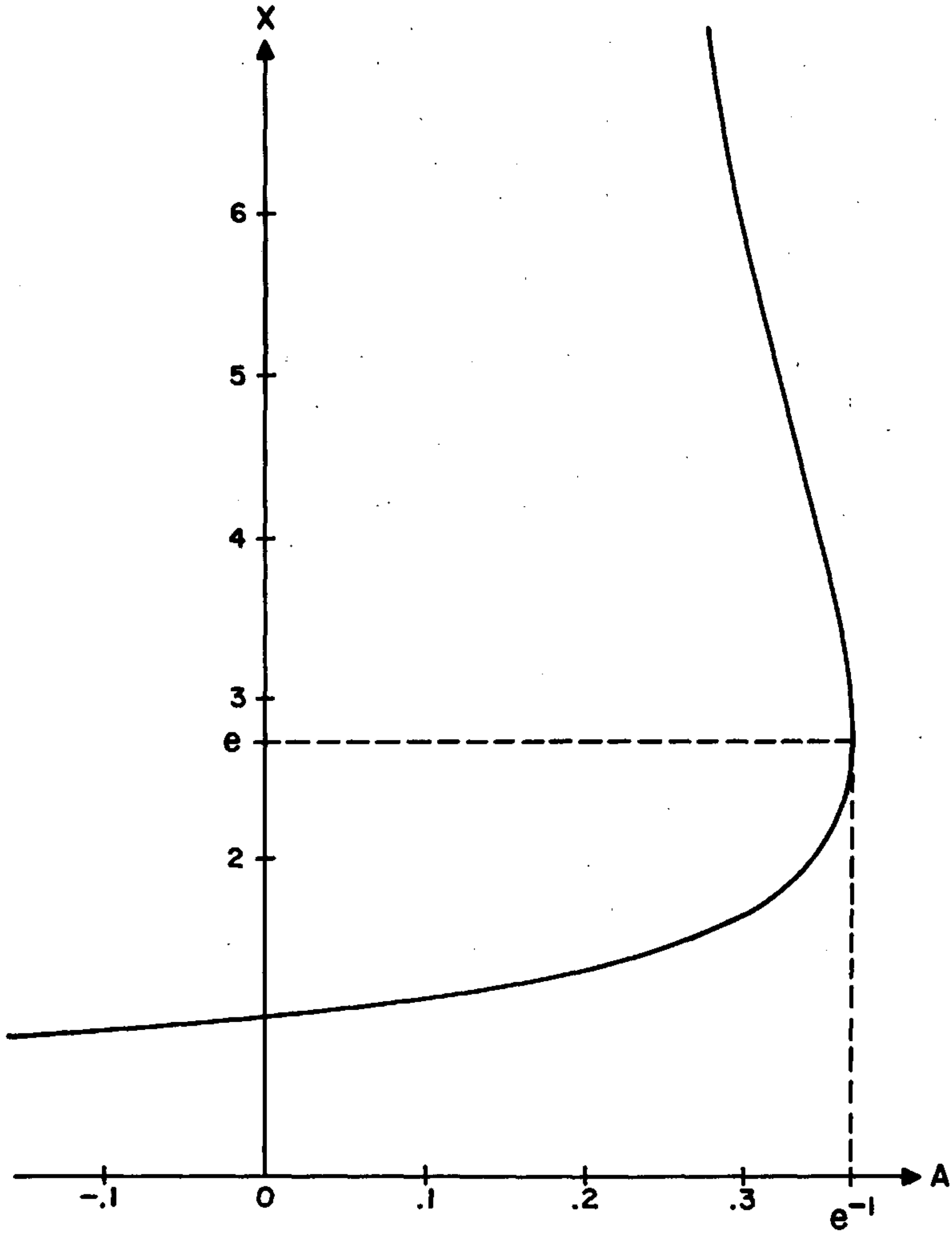

Figure 1. Solution of $X = e^{AX}$

TABLE 1. SOLUTIONS OF THE EQUATION $X = e^{AX}$

A	X_1	X_2
0.365	2.41047	3.09768
0.360	2.23913	3.39658
0.355	2.13025	3.63730
0.350	2.04754	3.85633
0.345	1.98002	4.06488
0.340	1.92263	4.26824
0.335	1.87257	4.46944
0.330	1.82808	4.67051
0.325	1.78801	4.87287
0.320	1.75153	5.07765
0.315	1.71804	5.28577
0.310	1.68707	5.49803
0.305	1.65826	5.71515
0.300	1.63134	5.93779
0.295	1.60607	6.16660
0.290	1.58226	6.40223
0.285	1.55976	6.64529
0.280	1.53842	6.89645
0.275	1.51815	7.15638
0.270	1.49883	7.42576
0.265	1.48039	7.70533
0.260	1.46274	7.99586
0.255	1.44584	8.29818
0.250	1.42961	8.61317
0.245	1.41402	8.94177
0.240	1.39901	9.28500
0.235	1.38454	9.64397
0.230	1.37058	10.01967
0.225	1.35710	10.41401
0.220	1.34406	10.82731
0.215	1.33144	11.26283
0.210	1.31921	11.72078
0.205	1.30736	12.20354
0.200	1.29586	12.71321
0.195	1.28469	13.25207
0.190	1.27383	13.82269
0.185	1.26327	14.42792
0.180	1.25300	15.07093
0.175	1.24300	15.75529
0.170	1.23325	16.48501
0.165	1.22375	17.26459

TABLE 1. SOLUTIONS OF THE EQUATION $X = e^{AX}$ (CONTINUED)

A	X_1	X_2
0.160	1.21448	18.09916
0.155	1.20544	18.99452
0.150	1.19661	19.95730
0.145	1.18798	20.99510
0.140	1.17956	22.11665
0.135	1.17132	23.33206
0.130	1.16326	24.65310
0.125	1.15537	26.09349
0.120	1.14765	27.66939
0.115	1.14009	29.39993
0.110	1.13269	31.30792
0.105	1.12544	33.42073
0.100	1.11833	35.77152
0.099	1.11692	36.27365
0.098	1.11552	36.78726
0.097	1.11413	37.31273
0.096	1.11274	37.85044
0.095	1.11135	38.40083
0.094	1.10998	38.96432
0.093	1.10860	39.54137
0.092	1.10723	40.13245
0.091	1.10587	40.73805
0.090	1.10452	41.35870
0.089	1.10316	41.99493
0.088	1.10182	42.64732
0.087	1.10047	43.31647
0.086	1.09914	44.00300
0.085	1.09781	44.70750
0.084	1.09648	45.43085
0.083	1.09516	46.17359
0.082	1.09384	46.93654
0.081	1.09253	47.72051
0.080	1.09122	48.52633
0.079	1.08992	49.35490
0.078	1.08862	50.20715
0.077	1.08733	51.08406
0.076	1.08604	51.98668
0.075	1.08476	52.91610
0.074	1.08348	53.87349
0.073	1.08221	54.86008
0.072	1.08094	55.87716

TABLE 1. SOLUTIONS OF THE EQUATION $X = e^{AX}$ (CONTINUED)

A	X_1	X_2
0.071	1.07967	56.92612
0.070	1.07841	58.00840
0.069	1.07716	59.12555
0.068	1.07590	60.27923
0.067	1.07466	61.47117
0.066	1.07342	62.70323
0.065	1.07218	63.97738
0.064	1.07094	65.29573
0.063	1.06971	66.66052
0.062	1.06849	68.07410
0.061	1.06727	69.53919
0.060	1.06605	71.05830
0.059	1.06484	72.63460
0.058	1.06363	74.27100
0.057	1.06243	75.97111
0.056	1.06123	77.73838
0.055	1.06004	79.57677
0.054	1.05884	81.49049
0.053	1.05766	83.48407
0.052	1.05647	85.56243
0.051	1.05529	87.73086
0.050	1.05412	89.99511
0.049	1.05295	92.36141
0.048	1.05178	94.83656
0.047	1.05062	97.42794
0.046	1.04946	100.14359
0.045	1.04830	102.99232
0.044	1.04715	105.98377
0.043	1.04601	109.12852
0.042	1.04486	112.43818
0.041	1.04372	115.92557
0.040	1.04259	119.60483
0.039	1.04145	123.49162
0.038	1.04032	127.60333
0.037	1.03920	131.95928
0.036	1.03808	136.58106
0.035	1.03696	141.49283
0.034	1.03585	146.72169
0.033	1.03474	152.29819
0.032	1.03363	158.25686
0.031	1.03253	164.63684

TABLE 1. SOLUTIONS OF THE EQUATION $X = e^{AX}$ (CONTINUED)

A	X_1	X_2
0.030	1.03143	171.48276
0.029	1.03033	178.84562
0.028	1.02924	186.78404
0.027	1.02815	195.36567
0.026	1.02706	204.66900
0.025	1.02598	214.78562
0.024	1.02490	225.82297
0.023	1.02383	237.90800
0.022	1.02276	251.19164
0.021	1.02169	265.85477
0.020	1.02062	282.11590
0.019	1.01956	300.24141
0.018	1.01850	320.55929
0.017	1.01745	343.47779
0.016	1.01640	369.51134
0.015	1.01535	399.31705
0.014	1.01430	433.74729
0.013	1.01326	473.92721
0.012	1.01222	521.37198
0.011	1.01119	578.16978
0.010	1.01015	647.27751
0.009	1.00912	733.01908
0.008	1.00810	841.96771
0.007	1.00707	984.60593
0.006	1.00605	1178.69368
0.005	1.00504	1456.79943
0.004	1.00402	1885.48511
0.003	1.00301	2624.17381
0.002	1.00201	4167.54073
0.001	1.00100	9118.00663
0.00	1.00000	
-.01	0.99015	
-.02	0.98058	
-.03	0.97128	
-.04	0.96224	
-.05	0.95345	
-.06	0.94488	
-.07	0.93654	
-.08	0.92842	
-.09	0.92049	
-.10	0.91277	

TABLE 1. SOLUTIONS OF THE EQUATION $X = e^{AX}$ (CONTINUED)

A	X_1	X_2
-.11	0.90522	
-.12	0.89786	
-.13	0.89067	
-.14	0.88364	
-.15	0.87677	
-.16	0.87005	
-.17	0.86347	
-.18	0.85704	
-.19	0.85075	
-.20	0.84458	
-.21	0.83854	
-.22	0.83262	
-.23	0.82682	
-.24	0.82113	
-.25	0.81555	
-.26	0.81008	
-.27	0.80471	
-.28	0.79944	
-.29	0.79427	
-.30	0.78918	
-.31	0.78419	
-.32	0.77929	
-.33	0.77447	
-.34	0.76973	
-.35	0.76508	
-.36	0.76050	
-.37	0.75600	
-.38	0.75157	
-.39	0.74721	
-.40	0.74292	
-.41	0.73870	
-.42	0.73454	
-.43	0.73045	
-.44	0.72642	
-.45	0.72245	
-.46	0.71854	
-.47	0.71469	
-.48	0.71090	
-.49	0.70715	
-.50	0.70347	
-.51	0.69983	

TABLE 1. SOLUTIONS OF THE EQUATION $X = e^{AX}$ (CONTINUED)

A	X_1	X_2
-.52	0.69625	
-.53	0.69271	
-.54	0.68923	
-.55	0.68579	
-.56	0.68240	
-.57	0.67905	
-.58	0.67575	
-.59	0.67249	
-.60	0.66927	
-.61	0.66610	
-.62	0.66296	
-.63	0.65987	
-.64	0.65681	
-.65	0.65379	
-.66	0.65081	
-.67	0.64787	
-.68	0.64496	
-.69	0.64208	
-.70	0.63924	
-.71	0.63644	
-.72	0.63366	
-.73	0.63092	
-.74	0.62821	
-.75	0.62553	
-.76	0.62289	
-.77	0.62027	
-.78	0.61768	
-.79	0.61512	
-.80	0.61259	
-.81	0.61008	
-.82	0.60760	
-.83	0.60515	
-.84	0.60273	
-.85	0.60033	
-.86	0.59795	
-.87	0.59561	
-.88	0.59328	
-.89	0.59098	
-.90	0.58870	
-.91	0.58645	
-.92	0.58422	

TABLE 1. SOLUTIONS OF THE EQUATION $X = e^{AX}$ (CONTINUED)

A	X_1	X_2
-.93	0.58201	
-.94	0.57982	
-.95	0.57766	
-.96	0.57551	
-.97	0.57339	
-.98	0.57129	
-.99	0.56921	
-1.00	0.56714	
-1.50	0.48391	
-2.00	0.42630	
-2.50	0.38343	
-3.00	0.34997	
-3.50	0.32294	
-4.00	0.30054	
-4.50	0.28161	
-5.00	0.26534	
-5.50	0.25119	
-6.00	0.23873	
-6.50	0.22767	
-7.00	0.21776	
-7.50	0.20883	
-8.00	0.20073	
-8.50	0.19333	
-9.00	0.18656	
-9.50	0.18032	
-10.00	0.17455	
-10.50	0.16920	
-11.00	0.16423	
-11.50	0.15958	
-12.00	0.15523	
-12.50	0.15116	
-13.00	0.14732	
-13.50	0.14370	
-14.00	0.14029	
-14.50	0.13706	
-15.00	0.13400	
-15.50	0.13109	
-16.00	0.12832	
-16.50	0.12569	
-17.00	0.12318	
-17.50	0.12079	

TABLE 1. SOLUTIONS OF THE EQUATION $X = e^{AX}$ (CONTINUED)

A	X_1	X_2
-18.00	0.11849	
-18.50	0.11630	
-19.00	0.11420	
-19.50	0.11218	
-20.00	0.11025	
-20.50	0.10839	
-21.00	0.10660	
-21.50	0.10488	
-22.00	0.10322	
-22.50	0.10162	
-23.00	0.10008	
-23.50	0.09859	
-24.00	0.09715	
-24.50	0.09575	
-25.00	0.09441	
-25.50	0.09310	
-26.00	0.09184	
-26.50	0.09061	
-27.00	0.08942	
-27.50	0.08827	
-28.00	0.08715	
-28.50	0.08606	
-29.00	0.08500	
-29.50	0.08397	
-30.00	0.08297	
-30.50	0.08200	
-31.00	0.08105	
-31.50	0.08013	
-32.00	0.07923	
-32.50	0.07835	
-33.00	0.07750	
-33.50	0.07667	
-34.00	0.07585	
-34.50	0.07506	
-35.00	0.07428	
-35.50	0.07352	

ACKNOWLEDGEMENT

The author gratefully acknowledges Dr. M. S. Taylor for his many useful editorial comments which were incorporated in this paper.

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2	Commander US Army Mobility Equipment Research & Development Command ATTN: Tech Docu Cen, Bldg. 315 DRSME-RZT Fort Belvoir, VA 22060
1	Commander US Army Armament Command Rock Island, IL 61202
1	Commander US Army Harry Diamond Labs ATTN: DRXDO-TI 2800 Powder Mill Road Adelphi, MD 20783
1	Director US Army TRADOC Systems Analysis Activity ATTN: ATAA-SA White Sands Missile Range NM 88002

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